ORC 67-10
SEPTEMBER 1967

SELECTION PROCEDURES FOR RESTRICTED FAMILIES OF PROBABILITY DISTRIBUTIONS

by

Richard E. Barlow and Shanti S. Gupta

OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING

UNIVERSITY OF CALIFORNIA - BERKELEY

31

SELECTION PROCEDURES FOR RESTRICTED FAMILIES OF PROBABILITY DISTRIBUTIONS

by

Richard E. Barlow
Operations Research Center
University of California, Berkeley

and

Shanti S. Gupta
Department of Statistics
Purdue University

September 1967 ORC 67-10

This research was partly supported by the Office of Naval Research contract NONR-1100(26) at Purdue University and Office of Naval Research contract NONR-3656(18) at the University of California at Berkeley. Reproduction in whole or in part is permitted for any purposes of the United States Government. This is also appearing as Technical Report #132 at the Statistics Department, Purdue University.

ABSTRACT

This paper is primarily concerned with selecting a subset of k populations such that the probability is at least P^* that the selected subset includes the population with the largest (smallest) quantile of a given order α $(0 < \alpha < 1)$. In particular a procedure is proposed and studied which is valid for any family of distributions with increasing failure rate on the average (IFRA). It is compared, asymptotically, with a distribution-free procedure proposed by Rizvi and Sobel.

SELECTION PROCEDURES FOR RESTRICTED FAMILIES OF PROBABILITY DISTRIBUTIONS[†]

by

Richard E. Barlow and Shanti S. Gupta

1. INTRODUCTION AND SUMMARY

Let $\Pi_1, \Pi_2, \ldots, \Pi_k$ be k populations. The random variable X_i associated with Π_i has a continuous distribution $F_i, i = 1,2, \ldots, k$. We are primarily interested in selecting a subset such that the probability is at least P^* that the selected subset includes the population with the largest (smallest) quantile of a given order $\alpha (0 < \alpha < 1)$. We assume each F_i has a unique α-quantile, $\xi_{\alpha,i}$. Let $F_{[i]}(x) = F_{[i]}$ denote the cumulative distribution function of the population with the ith smallest α-quantile. In the following, we consider families of distributions ordered in a certain sense with respect to a specified continuous distribution G and propose and study a selection procedure which is different from the nonparametric procedure of Rizvi and Sobel (1967). We assume

(a) $F_{[i]}(x) \geq F_{[k]}(x) \quad i = 1,2, \ldots, k$ and all x,

(b) $\exists$ a continuous distribution $G \ni F_{[i]} \preceq G, \forall i = 1,2, \ldots, k$,

where $\preceq$ denotes a partial ordering relation on the space of distributions.

[†]This research was partly supported by the Office of Naval Research contract NONR-1100(26) at Purdue University and Office of Naval Research contract NONR-3656(18) at the University of California at Berkeley. Reproduction in whole or in part is permitted for any purposes of the United States Government.

A relation $\prec$ on the space of distribution is a *partial ordering* if

$$F \prec F \;\forall \text{ distributions } F$$

$$F \prec G,\; G \prec H \text{ implies } F \prec H .$$

Note that $F \prec G$ and $G \prec F$ do *not* necessarily imply $F \equiv G$.

Various special cases in audition to stochastic ordering are:

$$F \underset{*}{\prec} G \text{ iff } F(0) = G(0) = 0 \text{ and } \frac{G^{-1}F(x)}{x} \text{ is} \tag{i}$$

nondecreasing in $x \geq 0$ on the support of F.

$$F \underset{c}{\prec} G \text{ iff } G^{-1}F(x) \text{ is convex on the support of } F . \tag{ii}$$

$$F \underset{r}{\prec} G \text{ iff } F(0) = G(0) = \tfrac{1}{2} \text{ and } \frac{G^{-1}F(x)}{x} \text{ is} \tag{iii}$$

increasing (decreasing) for x positive (negative) on the support of F.

$$F \underset{s}{\prec} G \text{ iff } F(0) = G(0) = \tfrac{1}{2} \text{ and } G^{-1}F \text{ is concave-convex} \tag{iv}$$

about the origin, on the support of F.

If $G(x) = 1 - e^{-x}$ for $x \geq 0$, then (i) defines the class of IFRA distributions studied by Birnbaum, Esary and Marshall (1966) while (ii) defines the class of IFR distributions studied by Barlow, Marshall and Proschan (1963). For any distribution G, $F \underset{*}{\prec} G$ iff $F(x)$ crosses $G(\theta x)$ at most once and from below if at all as a function of x for all $\theta > 0$. If $G(x) = 1 - \exp(-x^{\lambda})$ for $x \geq 0$ and $\lambda > 0$, then $F \underset{*}{\prec} G$ implies that F is "sharper" than the family of Weibull distributions with shape parameter λ. Implications of

orderings defined by (iii) were studied by Lawrence (1966). Van Zwet (1964) studied orderings defined by both (ii) and (iv). Clearly $\underset{c}{\prec}$ ordering implies $\underset{*}{\prec}$ ordering and $\underset{s}{\prec}$ ordering implies $\underset{r}{\prec}$ ordering.

If $\underset{\sim}{X}_i = (X_{i1}, X_{i2}, \ldots, X_{in})$ is the observed sample from the ith population, then we restrict ourselves to the class of statistics $T_i = T(\underset{\sim}{X}_i)$ that preserve both ordering relations (a) and (b), i.e.,

$$P_{F_{[i]}}\{T(\underset{\sim}{X}) \leq x\} \geq P_{F_{[k]}}\{T(\underset{\sim}{X}) \leq x\} \qquad (a')$$

for all x and $i = 1, 2, \ldots, k$.

$$F_{T(\underset{\sim}{X}_i)} \underset{-}{\prec} G_{T(\underset{\sim}{Y})} \qquad (b')$$

$i = 1, 2, \ldots, k$ where $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ is a random sample from G.

In Section 2 of this paper, we propose and study procedures R (R') for selecting the population with the largest (smallest) α-quantile for distributions which are $\underset{*}{\prec}$ ordered with respect to a specified distribution G. The infimum of the probability of a correct selection is obtained in Theorem 2.1 and asymptotic evaluation is given in Theorem 2.2. Section 3 deals with quantile selection procedures for the class of IFRA distributions. In Section 4, we study the efficiency of procedure R with respect to a procedure studied by Rizvi and Sobel (1967) under scale type slippage configurations. Asymptotic relative efficiency of R with respect to a selection procedure for the gamma populations proposed by Gupta (1963) is also investigated. Section 5 deals with selection procedures for the median for distributions that are $\underset{r}{\prec}$ ordered with respect to a specified G. In Section 6 we propose a selection procedure with respect to the sample means for distributions that are $\underset{c}{\prec}$ ordered with respect to $G(x) = 1 - e^{-x}$. Application to the selection of gamma populations is also given in Section 6.

2. QUANTILE SELECTION RULES FOR DISTRIBUTIONS $\underset{*}{\prec}$ ORDERED WITH RESPECT TO G

We are given a sample of size n from each of the k populations $\Pi_i, i = 1,2, \ldots, k$. The distributions $F_{[i]}$ and the specified distribution G satisfy the assumptions (a) and (b) of Section 1. The distributions F_i are, otherwise, unspecified. Of course, the correct pairing of the unordered and ordered F_i's is not known. We denote the k-tuples $(F_1, F_2, \ldots, F_k)$ by Ω . Let $T_{j,i}$ denote the jth order statistic from F_i where $j \le n\alpha < j + 1$. Clearly, $T_{j,i} \underset{a.s.}{\to} \xi_{\alpha,i}$, the α-quantile as $n \to \infty$ and $\frac{j}{n} \to \alpha$. The rule we propose for selecting the population with the largest α-quantile is:

R : Select population Π_i iff

$$T_{j,i} \ge c \max_{1 \le r \le k} T_{j,r} , \; j \le n\alpha < n + 1 , \tag{2.1}$$

where $c = c(k, P^*, n, j)$ is some number between 0 and 1 which is determined so as to satisfy the probability requirement

$$\inf_{\Omega} P\{CS|R\} = P^* , \tag{2.2}$$

where CS stands for a correct selection, i.e., the selection of any subset which contains the population $\Pi_{[k]}$ with distribution $F_{[k]}$. Before discussing the main theorem concerned with the evaluation of $P\{CS|R\}$, we present a known result for order statistics. Let $H_{j,i}(x)$ be the cdf of the jth order statistic from $F_{[i]}$ and let $G_j(x)$ be the cdf of the jth order statistic from G . Let us define

$$B_{j,n}(x) = \frac{n!}{(j-1)!(n-j)!} \int_0^x u^{j-1} (1 - u)^{n-j} \, du$$

so that

$$H_{j,i}(x) = B_{j,n}(F_{[i]}(x)) \equiv B_{j,n}F_{[i]}(x) . \tag{2.3}$$

Since

$$G_j^{-1} H_{j,i}(x) = [B_jG]^{-1} B_jF_{[i]}(x) = G^{-1} F_{[i]}(x) , \tag{2.4}$$

we see that order statistics preserve each of the partial ordering relations (i) - (iv). For additional applications of (2.4) see van Zwet (1964).

We now state and prove a theorem which enables us to compute the constant c which defines the procedure R .

Theorem 2.1:

If $F_{[i]}(0) = G(0) = 0$, $F_{[i]}(x) \geq F_{[k]}(x)$, $x \geq 0$, $i = 1,2, \ldots, k$ and $F_{[k]} \underset{*}{\prec} G$, then

$$\inf_{\Omega} P\{CS|R\} = \int_0^\infty [G_j(\tfrac{x}{c})]^{k-1} dG_j(x) . \tag{2.5}$$

Proof:

Note that

$$P\{CS|R\} = \int_0^\infty \left[\prod_{i=1}^{k-1} H_{j,i}(x/c)\right] dH_{j,k}(x)$$

$$\geq \int_0^\infty \left[H_{j,k}(x/c)\right]^{k-1} dH_{j,k}(x) .$$

We wish to bound the right-hand side. Let $X_{j,r} (r = 1,2, \ldots, k)$ be i.i.d. with c.d.f. $H_{j,k}(x)$. (Note that $X_{j,r} \equiv T_{j,r}$ when $F_{[i]} \equiv F_{[k]}, \forall_i$.) Let $\phi(x) = G_j^{-1} H_{j,k}(x) = G^{-1} F_{[k]}(x)$ so that $\phi(x)/x$ is nondecreasing in $x \geq 0$.
Then

$$\frac{\phi(X_{j,r})}{X_{j,r}} \leq \frac{\phi(\max_{1\leq r\leq k} X_{j,r})}{\max_{1\leq r\leq k} X_{j,r}} = \frac{\max_{1\leq r\leq k} \phi(X_{j,r})}{\max_{1\leq r\leq k} X_{j,r}} \tag{2.6}$$

so that

$$\frac{\phi(X_{j,r})}{\max_{1\leq r\leq k} \phi(X_{j,r})} \leq \frac{X_{j,r}}{\max_{1\leq r\leq k} X_{j,r}} . \tag{2.7}$$

Since $Y_{j,r} = \phi(X_{j,r})$ has distribution G_j for $r = 1,2, \ldots, k$, we have

$$P\{CS|R\} \geq P_{H_{j,k}}\left\{\frac{X_{j,r}}{\max_{1\leq r\leq k} X_{j,r}} \geq c\right\}$$

$$\geq P_{G_j}\left\{\frac{Y_{j,k}}{\max_{1\leq r\leq k} Y_{j,r}} \geq c\right\} \tag{2.8}$$

$$= \int_0^\infty [G_j(\tfrac{x}{c})]^{k-1}\, dG_j(x) ,$$

provided c is between 0 and 1. ||

<u>Remark 1</u>:

The constant $c = c(k, P^*, n, j)$ which defines the selection procedure R is determined by

$$\int_0^\infty [G_j(\tfrac{x}{c})]^{k-1}\, dG_j(x) = P^*, \; (\tfrac{1}{k} < P^* < 1) \;. \tag{2.9}$$

These constants are tabulated for $G(x) = 1 - e^{-x}$ for selected values of n, k, j and P^* in the first set of tables in the companion paper by Barlow, Gupta and Panchapakesan (1967). Clearly, c is independent of scale.

Remark 2:

If $G_j(x) = 1 - e^{-(\frac{x}{\theta})^\lambda}$, for $x \geq 0$ and $\theta, \lambda > 0$, then for $j = 1$, the values of c are independent of n. This can be seen from the fact that the distribution of the smallest order statistic involves n only as a scale parameter and that the selection procedure (2.1) is scale invariant.

Remark 3:

It should be pointed out that Theorem 2.1 requires only $F_{[k]} \underset{*}{\prec} G$; however, to apply the procedure R, we assume that $F_{[i]} \underset{*}{\prec} G, \forall_i$.

Now we discuss the asymptotic evaluation of the probability of a correct selection. We state and prove the following theorem.

Theorem 2.2:

If $F_{[k]}(x) \underset{*}{\prec} G$, $F_{[k]}(G)$ has a differentiable density $f_{[k]}(g)$ in a neighborhood of the α-quantile ξ_α (η_α) and $f_{[k]}(\xi_\alpha) \neq 0$ $(g(\eta_\alpha) \neq 0)$, then in our previous notation (see Theorem 2.1)

$$P_{H_{j,k}}\left\{\frac{X_{j,k}}{\max\limits_{1\leq r\leq k} X_{j,r}} \geq c\right\} \approx \int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{x}{c} + \frac{(1-c)\xi_\alpha\, f_k(\xi_\alpha)\sqrt{n}}{c\sqrt{\alpha\,\bar{\alpha}}}\right) d\Phi(x)$$
$$\geq \int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{x}{c} + \frac{(1-c)\eta_\alpha\, g(\eta_\alpha)\sqrt{n}}{c\sqrt{\alpha\,\bar{\alpha}}}\right) d\Phi(x) \tag{2.10}$$

as $n \to \infty$, where $\bar{\alpha} = 1 - \alpha$ and $\Phi(\cdot)$ is the c.d.f. of the standard normal variate.

Proof:

$$P\left\{X_{j,k} \geq c \max_{1 \leq r \leq k-1} X_{j,r}\right\}$$

$$= P\left\{\frac{(X_{j,k}-\xi_\alpha)\, f_{[k]}(\xi_\alpha)\sqrt{n}}{\sqrt{\alpha\,\bar{\alpha}}} \geq c\, \frac{\max\limits_{1 \leq r \leq k-1} (X_{j,r}-\xi_\alpha) + \frac{c-1}{c}\,\xi_\alpha}{\sqrt{\alpha\,\bar{\alpha}}/(\sqrt{n}\, f_{[k]}(\xi_\alpha))}\right\} \tag{2.11}$$

$$\approx \int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{x}{c} + \frac{(1-c)\,\xi_\alpha\, f_{[k]}(\xi_\alpha)\sqrt{n}}{c\sqrt{\alpha\,\bar{\alpha}}}\right) d\Phi(x) ,$$

since $$X_{j,k} \sim N\left(\xi_\alpha, \frac{\alpha\,\bar{\alpha}}{n\, f^2_{[k]}(\xi_\alpha)}\right) .$$

To prove the second part of (2.10), note that $F_{[k]} \underset{*}{<} G$ implies $G^{-1}F_{[k]}(x) - x$ changes sign at most once and from $-$ to $+$, if at all. Since $F_{[k]} \underset{*}{<} G$ is invariant under scale changes, we can assume $\eta_\alpha = \xi_\alpha$ so that $F(\xi_\alpha) \equiv G(\xi_\alpha)$. Either $F = G$ in a neighborhood of ξ_α or the slope of the tangent line to F at ξ_α is greater than the slope of the tangent line to G at ξ_α . In either case $f_{[k]}(\xi_\alpha) \geq g(\xi_\alpha)$ and in general $\xi_\alpha f_{[k]}(\xi_\alpha) \geq \eta_\alpha g(\eta_\alpha)$.

Remark 4:

Setting

$$\int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{x}{c} + \frac{(1-c)\,\eta_\alpha\, g(\eta_\alpha)\sqrt{n}}{c\sqrt{\alpha\,\bar{\alpha}}}\right) d\Phi(x) = P^* , \tag{2.12}$$

we see that

$$c(k, P^*, n, j) \approx 1 - \frac{\gamma(k, P^*, \alpha)}{\sqrt{n}} \tag{2.13}$$

as $n \to \infty$.

For $k = 2$, and $g(x) = e^{-x}$, we see

$$c(2, P^*, n, j) = 1 - \frac{2^{\frac{1}{2}}C}{n^{\frac{1}{2}}} + \frac{C}{n} - \frac{3}{2^{3/2}} \frac{C^3}{n^{3/2}} + O\left(\frac{1}{n^2}\right) \tag{2.14}$$

where
$$C = \frac{\Phi^{-1}(P^*)\sqrt{\alpha\,\bar{\alpha}}}{(1-\alpha)[-\log(1-\alpha)]} .$$

Subset Selection Rule for Smallest α-quantile

The rule for selecting the population with the smallest α-quantile is

R': Select population Π_i iff

$$dT_{j,i} \le \min_{1 \le i \le k} T_{j,i}, \quad j \le n < j+1 \tag{2.15}$$

where $0 < d = d(k, P^*, n, j) < 1$ is determined so as to satisfy the basic probability requirement. If $F_{[i]}(x) \le F_{[1]}(x)$, $i = 1,2, \ldots, k$ and all $x \ge 0$ and $F_{[1]} \underset{*}{\prec} G$, then the constant d is given by the equation

$$\int_0^\infty [\bar{G}_j(xd)]^{k-1} dG_j(x) = P^* \tag{2.16}$$

where $\bar{G}(x) = 1 - G(x)$. In a manner similar to the proof of Theorem 2.1 we can show that

$$P\{CS|R'\} \ge \int_0^\infty [\bar{G}_j(xd)]^{k-1} dG_j(x) .$$

The values of d are tabulated for selected values of k, P^*, n and j in the companion paper by Barlow, Gupta and Panchapakesan (1967).

The rules R and R' select nonempty subsets. The size of the selected subset is a random variable which takes values $1, 2, \ldots, k$. The expected size of the selected subset is a common measure of the efficiency of the procedure (Gupta (1963b)). However, it is difficult in our more general framework to set meaningful bounds on the expected size without further assumptions. If we assume, in addition, that there exists G^+ such that $G^+ \underset{*}{\prec} F_{[i]}$ for all i, then we can obtain an upper bound on the probability of including the "worst" population in the selected subset. We consider this in more detail later for IFRA distributions.

If we assume that $F_{[i]}(x)$ is stochastically increasing with respect to i, then

$$P\{\text{select } \Pi_{[i]}|R\} \geq P\{\text{select } \Pi_{[j]}|R\} \quad \text{if} \quad i \geq j . \tag{2.17}$$

The proof is similar to the one given in Gupta (1967). A result similar to (2.17) is true for R_1.

3. QUANTILE SELECTION PROCEDURES FOR THE CLASS OF IFRA DISTRIBUTIONS

If $F \underset{*}{\prec} G$ where $G(x) = 1 - e^{-x}$ for $x \geq 0$, then F is an IFRA distribution. The problem of selecting the best one of several IFRA populations has been considered by J. K. Patel (1967). He was interested in selecting that population with the smallest failure rate at a prescribed time T. His decision rule depends only on the number of observed failures in $[0, T]$ for each population and not on the times at which failure occurred. In this respect our procedure will utilize more information, though of course we are selecting with respect to quantiles rather than failure rate.

We show how to obtain c_λ values for the Weibull distribution with shape parameter $\lambda > 0$. We remark that the class of distributions F, such that

$$F \underset{*}{\prec} G_\lambda \quad \text{where} \quad G_\lambda(x) = 1 - e^{-\frac{x^\lambda}{\theta}} \quad \text{for } x \geq 0 \text{ and } \theta, \lambda > 0 \text{ is the smallest}$$

class of continuous distributions containing the Weibull class of distributions with shape parameter λ which is closed under the formation of coherent structures and limits in distribution.† To select populations $\underset{*}{\prec}$ ordered with respect to G_λ, choose c corresponding to n, k, j and P^* based on an exponential assumption and set $c_\lambda = (c)^{1/\lambda}$. To see this, let $Y_{j,i}$ denote the j-th order statistic from population i (all populations having the exponential distribution). Then $Y_{j,i}^{1/\lambda}$ is distributed as the j-th order statistic from G_λ and

$$\begin{aligned} P_{G_\lambda}\{CS|R\} &= P\left\{ \frac{Y_{j,i}^{1/\lambda}}{\max\limits_{1\leq i\leq k} Y_{j,i}^{1/\lambda}} \geq c_\lambda \right\} \\ &= P\left\{ \frac{Y_{j,i}}{\max\limits_{1\leq i\leq k} Y_{j,i}} \geq (c_\lambda)^\lambda \right\} \end{aligned} \tag{3.1}$$

†Private communication with James Esary.

or
$$c = (c_\lambda)^\lambda .$$

If, in addition to the assumptions of Section 2 (see Theorem 2.1), we assume that a) $F_{[1]}(x) \geq F_{[i]}(x)$ for all $x \geq 0$, $i = 1,2, \ldots, k$ and, b) $G_\lambda \underset{*}{<} F_{[1]}$ for all $i = 1,2, \ldots, k, \lambda > 1$ then we can obtain an upper bound on the probability of selecting the "worst" population, i.e.,

$$P\{\text{Selecting } \Pi_{[1]} | R\} \leq \int_0^\infty [G_j(\frac{x}{c^\lambda})]^{k-1} \, dG_j(x) \tag{3.2}$$

where c is chosen so that

$$P\{CS|R\} \geq \int_0^\infty [G_j(\frac{x}{c})]^{k-1} \, dG_j(x) = P^* .$$

Clearly, the upper bound is an increasing function of λ for $\lambda \geq 1$.

Now we describe the relative performance for small sample size $(n = 15)$ for procedures R and R_1 (the Rizvi-Sobel procedure - see (4.8)) using Monte Carlo techniques. For this purpose we chose gamma and Weibull distributions with densities

$$\text{gamma,} \quad \frac{e^{-x/\theta_i}}{\theta_i \Gamma(r)} \left(\frac{x}{\theta_i}\right)^{r-1}, \quad i = 1,2$$

$$\text{Weibull,} \quad e^{-(x/\theta_i)^r} \frac{r}{\theta_i} \left(\frac{x}{\theta_i}\right)^{r-1}, \quad i = 1,2 .$$

Based on 5,000 simulations, we computed $P\{CS|R\}$, $E(S|R)$ and $P\{CS|R_1\}$, $E(S|R_1)$. ($E(S|R)$ is the expected size of the selected subset using procedure R .) These values are given below:

Monte Carlo Comparisons[+] of R and R_1

$P^* = .90, \quad k = 2, \quad n = 15$

	Gamma $r = 1$ $\theta_1 = 1, \quad \theta_2 = 2$	Gamma $r = 5$ $\theta_1 = 2, \quad \theta_2 = 3$	Weibull $r = 2$ $\theta_1 = 1, \quad \theta_2 = 2$
$P\{CS\|R\}$	.993	1.000	.985
$E(S\|R)$	1.47	1.88	1.85
$P\{CS\|R_1\}$	.997	1.000	.939
$E(S\|R_1)$	1.64	1.96	1.76

[+] Computations performed by David Stanford of the Operations Research Center, University of California, Berkeley.

4. EFFICIENCY OF PROCEDURE R UNDER SLIPPAGE CONFIGURATIONS

We consider slippage configurations $F_{[i]}(x) = F(\frac{x}{\delta})$, $i = 1,2, \ldots, k-1$ and $F_{[k]}(x) = F(x)$, $0 < \delta < 1$. We obtain asymptotic expressions for the probability of a correct selection and the expected size of the selected subset for procedure R and for two other procedures.

Using our previous notation

$$P\{CS|R\} = P\left\{T_{j,k} \geq c \max_{1\leq r\leq k} T_{j,r}\right\}$$

$$= P_{H_{j,k}}\left\{X_{j,k} \geq c \max_{1\leq r\leq k} \delta X_{j,r}\right\} \tag{4.1}$$

$$\geq P_{G_j}\left\{Y_{j,k} \geq c\delta \max_{1\leq r\leq k} Y_{j,r}\right\}$$

where $Y_{j,r}$, $r = 1,2, \ldots, k$ are i.i.d. with the cdf $G_j(y)$. From (2.11), we obtain

$$P\{CS|R\} \approx \int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{x}{c\delta} + \frac{(1-c\delta)\xi_\alpha f(\xi_\alpha)\sqrt{n}}{c\delta\sqrt{\alpha\bar{\alpha}}}\right) d\Phi(x) . \tag{4.2}$$

Note that the probability of a correct selection is a monotone decreasing function of δ. For the slippage configuration

$$E(S|R) = P\{CS|R\} + (k-1)\, P\left\{T_{j,1} \geq c \max_{i\neq 1} T_{j,i}\right\} \tag{4.3}$$

$$P\left\{T_{j,1} \geq c \max_{i\neq 1} T_{j,i}\right\}$$

$$= P\left\{T_{j,1}-\delta\xi_\alpha \geq c \max\left(\max_{2\leq i\leq k-1} (T_{j,i}-\delta\xi_\alpha), T_{j,k}-\xi_\alpha+\xi_\alpha-\delta\xi_\alpha\right)+c\delta\xi_\alpha-\delta\xi_\alpha\right\} \tag{4.4}$$

$$\approx \int_{-\infty}^{\infty} \Phi\left(\frac{\delta x}{c} - \frac{f(\xi_\alpha)\xi_\alpha(1-\frac{\delta}{c})\sqrt{n}}{\sqrt{\alpha\bar{\alpha}}}\right) \Phi^{k-2}\left(\frac{x}{c} - \frac{\xi_\alpha f(\xi_\alpha)(1-\frac{1}{c})\sqrt{n}}{\sqrt{\alpha\bar{\alpha}}}\right) d\Phi(x) .$$

Setting $k = 2$, we have

$$E(S|R) - P\{CS|R\} \approx \Phi\left(-\frac{f(\xi_\alpha)\ \xi_\alpha(1-\frac{\delta}{c})\sqrt{n}}{\sqrt{\alpha\ \bar{\alpha}}\ \sqrt{1+(\frac{\delta}{c})^2}}\right). \tag{4.5}$$

Setting the right-hand side of (4.5) equal to ε, $-f(\xi_\alpha)\ \xi_\alpha\left(1-\frac{\delta}{c_n}\right)\sqrt{n} = \sqrt{\alpha\ \bar{\alpha}}\sqrt{1+\left(\frac{\delta}{c_n}\right)^2}\ \Phi^{-1}(\varepsilon)$, where we have put $c = c_n$. Now using $c_n \approx 1 - \frac{C}{\sqrt{n}}$ (from (2.14))

$$-f(\xi_\alpha)\ \xi_\alpha\left(1-\delta-\frac{\sqrt{2}C\delta}{\sqrt{n}}\ \sqrt{n}\right) = \Phi^{-1}(\varepsilon)\sqrt{\alpha\ \bar{\alpha}}\sqrt{1+\delta^2\left(1+\frac{2\sqrt{2}C}{\sqrt{n}}+\frac{2C^2}{n}\right)} \tag{4.6}$$

from which, keeping terms of order $\sqrt{n}$, we obtain

$$n_R(\varepsilon) \approx \frac{[-\sqrt{\alpha\ \bar{\alpha}}\ \Phi^{-1}(\varepsilon)(1+\delta^2)^{\frac{1}{2}} + \sqrt{2}C\delta\xi_\alpha\ f(\xi_\alpha)]^2}{\xi_\alpha^2\ f^2(\xi_\alpha)(1-\delta)^2}\ . \tag{4.7}$$

Comparison with Rizvi - Sobel Procedure

Rizvi and Sobel (1967) propose and investigate a distribution-free procedure, R_1, for the quantile selection problem.

R_1 : Select population Π_i iff

$$T_{j,i} \geq \max_{1\leq r\leq k} T_{j-a,r} \tag{4.8}$$

where a is the smallest integer with $1 \leq a \leq j-1$ for which

$$\inf_{\Omega} P\{CS|R_1\} \geq P^* \tag{4.9}$$

is satisfied.

A disadvantage of this procedure is that for any given α and k a value

of $a \leq j-1$ may not exist for some pairs (n, P^*). However if P^* is chosen not greater than some function $P_1(n, \alpha, k)$ where $1/k < P_1 < 1$, then a value of $a \leq j-1$ does exist that satisfies (4.9). Rizvi and Sobel compare the efficiency of this procedure relative to several competing procedures under translation configurations.

We discuss the asymptotic probability of a correct selection using their procedure under the slippage configuration.

$$P\{CS|R_1\} = P\left\{T_{j,k} \geq \max_{1\leq r\leq k-1} T_{j-a,r}\right\}$$

$$= P\left\{\frac{(T_{j,k}-\xi_\alpha)\sqrt{n}\, f(\xi_\alpha)}{\sqrt{\alpha\,\bar{\alpha}}} \geq \max_{1\leq r\leq k-1}\left(\frac{T_{j-a,r}-\delta\xi_\alpha+(\delta-1)\xi_\alpha}{\delta\sqrt{\alpha\,\bar{\alpha}}}\right)\delta\sqrt{n}\, f(\xi_\alpha)\right\} \tag{4.10}$$

$$\approx \int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{x}{\delta} + \frac{\gamma}{\sqrt{\alpha\,\bar{\alpha}}} - \frac{(1-\delta)\xi_\alpha\sqrt{n}\, f(\xi_\alpha)}{\delta\sqrt{\alpha\,\bar{\alpha}}}\right) d\Phi(x)\ .$$

The derivation above uses the fact that

$$\frac{T_{j-a,r}-\delta\xi_\alpha}{\sqrt{\alpha\,\bar{\alpha}}}\sqrt{n}\,\frac{1}{\delta}\, f(\xi_\alpha) \xrightarrow[\text{law}]{} N\left(\frac{-\gamma}{\sqrt{\alpha\,\bar{\alpha}}},\ 1\right)$$

where $\gamma/\sqrt{n} = a/(n+1)$. (See Lemma 2 of Rizvi and Sobel (1967).)

Similarly

$E(S|R_1) - P\{CS|R_1\} \approx$

$$\int_{-\infty}^{\infty} \phi\left(\delta x + \frac{\gamma}{\sqrt{\alpha\,\bar{\alpha}}} - \frac{(1-\delta)\xi_\alpha\sqrt{n}\, f(\xi_\alpha)}{\sqrt{\alpha\,\bar{\alpha}}}\right)\Phi^{k-2}\left(x + \frac{\gamma}{\sqrt{\alpha\,\bar{\alpha}}}\right) d\Phi(x)\ . \tag{4.11}$$

Setting $k = 2$, we have from (4.11)

$$E(S|R_1) - P\{CS|R_1\} \approx \Phi\left(\frac{\gamma(1-\delta)\ \xi_\alpha + f(\xi_\alpha)\sqrt{n}}{\sqrt{\alpha\ \bar{\alpha}}\ \sqrt{1+\delta^2}}\right) . \tag{4.12}$$

Equating the right-hand side of (4.12) to ε , we obtain

$$n_{R_1}(\varepsilon) \approx \frac{[\sqrt{\alpha\ \bar{\alpha}}\sqrt{1+\delta^2}\ \Phi^{-1}(\varepsilon)-\gamma]^2}{(1-\delta)^2[\xi_\alpha\ f(\xi_\alpha)]^2} . \tag{4.13}$$

For the slippage configuration above we define the asymptotic relative efficiency A R E$(R,R_1;\delta)$ of R relative to R_1 to be the limit as $\varepsilon \to 0$ of the ratio of $n_{R_1}(\varepsilon)$ to $n_R(\varepsilon)$.

$$\begin{aligned} \text{A R E}(R,R_1;\delta) &= \lim_{\varepsilon \downarrow 0} \frac{n_{R_1}(\varepsilon)}{n_R(\varepsilon)} \\ &= \lim_{\varepsilon \downarrow 0} \frac{[\sqrt{\alpha\ \bar{\alpha}}\ \sqrt{1+\delta^2}\ \Phi^{-1}(\varepsilon) - \gamma]^2}{\left[\sqrt{\alpha\ \bar{\alpha}}\ \Phi^{-1}(\varepsilon)\sqrt{(1+\delta^2)} + \sqrt{2}\delta C\xi_\alpha\ f(\xi_\alpha)\right]^2} \qquad (4.14) \\ &= 1 . \end{aligned}$$

Using the fact that $F \underset{*}{\prec} G$ implies

$\xi_\alpha\ f(\xi_\alpha) \geq \eta_\alpha\ g(\eta_\alpha)$, we see that

$$P_F\{CS|R_1\} \geq P_G\{CS|R_1\} \tag{4.15}$$

and

$$E_F(S|R_1) - P_F\{CS|R_1\} \leq E_G(S|R_1) - P_G\{CS|R_1\} \tag{4.16}$$

where both (4.15) and (4.16) are asymptotically true as $n \to \infty$ for the slippage configuration.

Comparison with Gupta Procedure

Gupta (1963) gave a selection procedure for gamma populations with densities

$$\frac{1}{\Gamma(r)\theta_i} \exp\left(-\frac{x}{\theta_i}\right) \left(\frac{x}{\theta_i}\right)^{r-1},\ x > 0,\ \theta_i > 0,\ i = 1,2,\ \ldots,\ k\ .$$

This procedure, R_2 , based on the means of sample size n from each of the k populations is:

R_2: Select the population corresponding to the observed mean $\bar{x}_i$ iff

$$\bar{x}_i \geq b \max_{1 \leq j \leq k} \bar{x}_j \tag{4.17}$$

where b is the largest constant $(0 < b \leq 1)$ chosen so that $P\{CS|R_2\} \geq P^*$. Letting $\nu = 2n\,r$, it is shown that $\log_e b \approx -d\sqrt{\frac{2}{\nu-1}}$ where d is independent of n and satisfies

$$\int_{-\infty}^{\infty} \Phi^{k-1}(x + d)\, d\Phi(x) = P^*\ . \tag{4.18}$$

Assume that the ranked θ_i's have the slippage configuration $\theta_{[i]} = \delta\theta_{[k]}$, $0 < \delta < 1,\ i = 1,2,\ \ldots,\ k - 1$. Then

$$E(S|R_2) - P\{CS|R_2\} \approx (k - 1) \int_{-\infty}^{\infty} \Phi^{k-2}\left(x - \frac{\log b}{\sqrt{2/(\nu-1)}}\right) \Phi\left(x - \frac{(\log b/\delta)}{\sqrt{2/(\nu-1)}}\right) d\Phi(x) \tag{4.19}$$

so that for $k = 2$

$$E(S|R_2) - P(CS|R_2) \quad \Phi\left(\frac{-\log (b/\delta)}{\sqrt{\frac{2}{\nu-1}}\sqrt{2}}\right) . \tag{4.20}$$

Setting the right-hand side of (4.20) equal to ϵ and solving for $n = n_{R_2}(\epsilon)$

$$n_{R_2}(\epsilon) \approx \frac{[2\Phi^{-1}(\epsilon) - \sqrt{2}]^2}{2r(\log \delta)^2} . \tag{4.21}$$

$$\begin{aligned} A\ R\ E(R,R_2;\delta) &\approx \lim_{\epsilon \downarrow 0} \frac{n_{R_2}(\epsilon)}{n_R(\epsilon)} \\ &= \frac{2(1-\delta)^2[\xi_\alpha\ f(\xi_\alpha)]^2}{r[\log \delta]^2\ \alpha\ \bar{\alpha}(1+\delta^2)} . \end{aligned} \tag{4.22}$$

for $r \geq 1$, $\quad A\ R\ E(R,R_2;\delta) \geq \dfrac{2(1-\alpha)^2\ (1-\alpha)^2\ [-\log\ (1-\alpha)]^2}{r(\log \delta)^2\ \alpha\ \bar{\alpha}(1+\delta^2)}$.

$$\text{Also} \quad \lim_{\delta \uparrow 1}\ A\ R\ E(R,R_2;\delta) = \frac{[\xi_\alpha\ f(\xi_\alpha)]^2}{r\ \alpha\ \bar{\alpha}} \tag{4.23}$$

so that $\quad \lim\limits_{\delta \uparrow 1}\ A\ R\ E(R,R_2;\delta) \geq \dfrac{(1-\alpha)^2\ (-\log(1-\alpha)^2}{\alpha(1-\alpha)}$, (letting $r = 1$) .

For $\alpha = \frac{1}{2}$ and $r = 1$, $\lim\limits_{\delta \uparrow}\ A\ R\ E(R,R_2;\delta) = [\log\ 2]^2 = .493$.

5. SELECTION WITH RESPECT TO THE MEDIAN FOR DISTRIBUTIONS r-ORDERED WITH RESPECT TO A SPECIFIED DISTRIBUTION G

We consider selection procedures with respect to the median for distributions F_i which have lighter tails than a specified distribution G. We say that F_i has lighter tails than G if F_i centered at its median, Δ_i, is $\underset{r}{\prec}$- ordered with respect to G $(G(0) = \frac{1}{2})$ and $\frac{d}{dx} F_i(x + \Delta_i)\big|_{x=0} \geq \frac{d}{dx} G(x)\big|_{x=0}$. Here we are following an ordering proposed by Doksum (1967). (See van Zwet (1964), Lawrence (1966).)

We wish to select a subset of the k populations containing the population with the largest median $\Delta_{[k]}$. The selection rule, we propose, is in terms of the sample medians. We use the same notation as in Section 2. The rule, R_3, is:

R_3 : Select Π_i iff

$$T_{j,i} \geq \max_{1 \leq r \leq k} T_{j,r} - D, \; j \leq \frac{n}{2} < j + 1 \tag{5.1}$$

and D is chosen to satisfy

$$\inf_{\Omega_1} P\{CS|R_3\} = P^* \tag{5.2}$$

where Ω_1 is set of all k-tuples $(F_1, F_2, \ldots, F_k)$ satisfying assumptions given above.

Now, we state and prove a theorem related to the infimum of probability of a correct selection when rule R_3 is used. Let $F_{[i]}(x)$ denote the distributions with median $\Delta_{[i]}$, $i = 1, 2, \ldots, k$.

Theorem 5.1:

If $F_{[i]}(x) \geq F_{[k]}(x)$, for all x, $G(0) = \frac{1}{2}$ and $G^{-1}F_{[k]}(x + \Delta_{[k]})$ is nondecreasing (nonincreasing) in $x \geq 0$ $(x \leq 0)$ and

$$\frac{d}{dx} F_{[k]}(x + \Delta_{[k]})\Big|_{x=0} \geq \frac{d}{dx} G(x)\Big|_{x=0} > 0 ,$$

then
$$\inf_{\Omega_1} P\{CS|R_3\} = \int_{-\infty}^{\infty} G_j^{k-1} (t + D)\, dG_j(t)$$

where G_j is defined as before.

Proof:

By stochastic ordering of the order statistics, we have

$$P\{CS|R_3\} \geq \int_{-\infty}^{\infty} H_{j,k}^{k-1} (t + D)\, dH_{j,k}(x) \tag{5.3}$$

$$= P\left\{X_{j,k} \geq \max_{1\leq r\leq k-1} X_{j,r} - D\right\}$$

$$= P\left\{X_{j,k} - \Delta_{[k]} \geq \max_{1\leq r\leq k-1} (X_{j,r} - \Delta_{[k]}) - D\right\}$$

where $X_{j,1}, X_{j,2}, \ldots, X_{j,k}$ are i.i.d.r.v. with distribution $H_{j,k}(x) \equiv B_{j,n} F_{[k]}(x)$.

Let $\phi(x) = G^{-1}F_{[k]} (x + \Delta_{[k]}) = G_j^{-1} M(x)$ where $M \equiv B_{j,n} F_{[k]} (x + \Delta_{[k]})$ is the distribution of $X_{j,r} - \Delta_{[k]}$. Note that $\phi(X_{j,r} - \Delta_{[k]}) = Y_{j,r}$ has distribution G_j. Now $\frac{\phi(x)}{x} \uparrow$ in $x \geq 0$, $\frac{\phi(x)}{x} \downarrow$ in $x \leq 0$ and $\phi'(0) > 1$ imply

$$\frac{\phi\left(\max_{1<r<k-1} (X_{j,r} - \Delta_{[k]})\right) - \phi(X_{j,k} - \Delta_{[k]})}{\max_{1\leq r\leq k-1} (X_{j,r} - \Delta_{[k]}) - (X_{j,k} - \Delta_{[k]})} \geq 1 \quad . \tag{5.4}$$

To see (5.4) we need only verify that

$$\frac{\phi(y) - \phi(x)}{y - x} \geq 1 \quad \text{for} \quad y \geq x \quad .$$

Note that $\phi'(0) = \frac{f_{[k]}(\Delta_{[k]})}{g(0)} \geq 1$ where $f_{[k]}$ (g) is the density of $F_{[k]}$ (G) .

Case 1:

$$(0 < x < y) \ \frac{\phi(x)}{x} \uparrow \ \text{in} \ x \geq 0 \ \text{and} \ \phi'(0) \geq 1$$

imply
$$\frac{\phi(y) - \phi(x)}{y - x} - \frac{\phi(x)}{x} \geq 1 \ .$$

Case 2:

$(x < y < 0)$ follows from $\frac{\phi(x)}{x} \downarrow$ in $x \leq 0$ and $\phi'(0) \geq 1$.

Case 3:

$(x < 0 < y)$. In this case $\phi(x) \leq x$ and $\phi(y) \geq y$

imply
$$\frac{\phi(y) - \phi(x)}{y - x} \geq 1 \ .$$

We have, by (5.4)

$$\max_{1 \leq r \leq k-1} Y_{j,r} - Y_{j,k} \geq \max_{1 \leq r \leq k-1} X_{j,r} - X_{j,k} \tag{5.5}$$

which implies

$$P\left\{Y_{j,k} \geq \max_{1 \leq r \leq k-1} Y_{j,r} - D\right\} \leq P\left\{X_{j,k} \geq \max_{1 \leq r \leq k-1} X_{j,r} - D\right\} \tag{5.6}$$

which proves the result.

Note that we need not have restricted attention only to medians.

6. SELECTION WITH RESPECT TO THE MEANS

Let μ_i be the mean of the distribution $F(x;\mu_i)$, $i = 1,2, \ldots, k$ and assume

(a) $F(x;\mu_{[i]}) \geq F(x;\mu_{[k]})$ for $i = 1,2, \ldots, k-1$ and all x .

(b) $F(x;\mu_{[i]}) \underset{c}{\prec} G(x) = 1 - e^{-x}$ for $i = 1,2, \ldots, k$.

Note that by assumption (b) we are confining attention to the so-called IFR class of distributions. It will also be convenient to assume $F(0;\mu_{[i]}) = 0$ for all i .

Let $\bar{x}_i = \sum_{j=1}^{n} x_{ij}/n$, where x_{ij} is the j-th observation in a random sample of size n from Π_i . Let $K_i(x) = K(x;\mu_i)$ be the distribution of $\bar{x}_i$. Then if $K_{[i]}(x) = K(x;\mu_{[i]})$

$$K_{[i]}(x) \geq K_{[k]}(x) \quad \text{for } i = 1,2, \ldots, k-1 \text{ and all } x . \tag{6.1}$$

$$K_{[i]} \underset{c}{\prec} G \quad \text{for } i = 1,2, \ldots, k . \tag{6.2}$$

(6.1) is an immediate consequence of (a) while (5.2) follows from (b) and the closure of IFR distributions under convolution (see Barlow, Marshall and Proschan (1963)).

If we are interested in selecting a subset containing the population with the largest $\mu_{[k]}$, we use the rule

R_4 : Select population Π_i iff

$$\bar{x}_i \geq c' \max_{1 \leq i \leq k} \bar{x}_i .$$

It follows that

Theorem 6.1:

$$P\{CS|R_4\} \geq \int_0^\infty [G(\frac{x}{c'})]^{k-1} \, dG(x)$$

where $G(x) = 1 - e^{-x}$.

The proof is the same as for Theorem 2.1. The disadvantage is that the right-hand side of the inequality is independent of n . However, by restricting the class of distributions to the gamma family we can obtain a lower bound which depends on n .

Application to the Selection of gamma Populations

Let us consider k populations with densities

$$\lambda_i^\alpha x^{\alpha-1} e^{-\lambda_i x} / \Gamma(\alpha), \ x \geq 0, \ \lambda_i > 0, \ i = 1,2, \ldots, k .$$

Assume that $\alpha \geq 1$, but otherwise unknown. This implies that the distributions are IFR. We are interested in selecting the population with the largest (smallest) value, $\lambda_{[k]}$ $(\lambda_{[1]})$, based on an independent sample of size n from each of the k populations. Note that $\mu_{[i]} = \alpha/\lambda_{[i]}$ for $i = 1,2, \ldots, k$. The subset selection rule based on the sample means, $\bar{x}_i$, $i = 1,2, \ldots, k$, is R_4 as before.

Let $G^{(\alpha)}$ denote a gamma distribution with parameter α . Since $\sum_{j=1}^{n} x_{ij}$ is distributed as a gamma random variable with distribution, $G^{(n\alpha)}$ it follows from a result of van Zwet (1964) that

$$G^{(n\alpha)} \underset{c}{\prec} G^{(n)}$$

when $\alpha \geq 1$. It follows that in this case

$$P\{CS|R_4\} > \int_0^\infty [G^{(n)}(\frac{x}{c'})]^{k-1} dG^{(n)}(x) . \tag{6.3}$$

The constant c' is determined by

$$\int_0^\infty [G^{(n)}(\frac{x}{c'})]^{k-1} dG^{(n)}(x) = P^* . \tag{6.4}$$

The values of c' are tabulated in Gupta (1963). It should be pointed out that for selecting the population with the smallest λ , the rule can be modified to:

R_5 : Select population Π_i iff

$$\bar{x}_i \leq d \min_{1\leq i\leq k} \bar{x}_i$$

where d is determined by

$$\int_0^\infty [1 - G^{(n)}(\frac{x}{d})]^{k-1} dG^{(n)}(x) = P^* .$$

The values of d are tabulated in Gupta and Sobel (1962).

The shape parameter $\alpha \geq 1$ need not be the same for all populations. It is only necessary that the distribution of the population, $\Pi_{[k]}$, with the largest mean be stochastically larger than the others.

REFERENCES

Barlow, R. E., S. S. Gupta, and S. Panchapakesan, "On the Distribution of the Maximum and Minimum of Ratios of Order Statistics," Mimeograph Series #133, Department of Statistics, Purdue University, (1967).

Barlow, R. E., A. W. Marshall, and F. Proschan, "Properties of Probability Distributions with Monotone Hazard Rate," Ann. Math. Statist., Vol. 34, pp. 375-389, (1963).

Birnbaum, Z. W., J. D. Esary, and A. W. Marshall, "A Stochastic Characterization of Wear-Out for Components and Systems," Ann. Math. Statist., Vol. 37, pp. 816-825, (1966).

Doksum, K., "Starshaped Transformations and Power of Rank Tests," Presented at the IMS Meeting at Washington, D.C., (1967).

Gupta, S. S., "Probability Integrals of the Multivariate Normal and Multivariate t ," Ann. Math. Statist., Vol. 34, pp. 792-828, (1963a).

Gupta, S. S., "On a Selection and Ranking Procedure for Gamma Populations," Ann. Inst. Statist. Math., Vol. 14, pp. 199-216, (1963b).

Gupta, S. S., ON SOME SELECTION AND RANKING PROCEDURES FOR MULTIVARIATE NORMAL POPULATIONS USING DISTANCE FUNCTIONS, MULTIVARIATE ANALYSIS, Ed. P. R. Krishnaiah, Academic Press, New York, pp. 457-475, (1966).

Gupta, S. S., and M. Sobel, "On Selecting a Subset Containing the Population with the Smallest Variance," Biometrika, Vol. 49, pp. 495-507, (1962).

Lawrence, M. J., "Inequalities and Tolerance Limits for s-Ordered Distributions," Technical Report ORC 66-37, Operations Research Center, College of Engineering, University of California, Berkeley, (1966).

Patel, J. K., "Selecting a Subset Containing the Best One of Several IFRA Populations (Abstract)," Ann. Math. Statist., Vol. 38, p. 957, (1967).

Rizvi, M. H., and M. Sobel, "Nonparametric Procedures for Selecting a Subset Containing the Population with the Largest α-Quantile," Ann. Math. Statist., Vol. 38, pp. 1788-1802, (1967).

van Zwet, W. R., CONVEX TRANSFORMATIONS OF RANDOM VARIABLES, Mathematical Centre, Amsterdam, The Netherlands, (1964).

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author) University of California, Berkeley	2a REPORT SECURITY CLASSIFICATION Unclassified 2b GROUP

3 REPORT TITLE

SELECTION PROCEDURES FOR RESTRICTED FAMILIES OF PROBABILITY DISTRIBUTIONS

4 DESCRIPTIVE NOTES (Type of report and inclusive dates)

Research Report

5 AUTHOR(S) (Last name, first name, initial)

BARLOW, Richard E.
GUPTA, Shanti S.

6 REPORT DATE September 1967	7a TOTAL NO OF PAGES 26	7b NO OF REFS 12

8a CONTRACT OR GRANT NO Nonr-3656(18) b PROJECT NO NR 042 238 c Research Project No.: WW 041 d	9a ORIGINATOR'S REPORT NUMBER(S) ORC 67-10 9b OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10 AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11 SUPPLEMENTARY NOTES Also supported by the Office of Naval Research under Contract Nonr-1100(26) at Purdue University.	12 SPONSORING MILITARY ACTIVITY MATHEMATICAL SCIENCE DIVISION

13 ABSTRACT

This paper is primarily concerned with selecting a subset of k populations such that the probability is at least P^* that the selected subset includes the population with the largest (smallest) quantile of a given order α $(0 < \alpha < 1)$. In particular a procedure is proposed and studied which is valid for any family of distributions with increasing failure rate on the average (IFRA). It is compared, asymptotically, with a distribution-free procedure proposed by Rizvi and Sobel.

DD FORM 1473

Unclassified
Security Classification

Unclassified
Security Classification

14 KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Selection Procedures Increasing Failure Rate Life Testing Quantiles						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.